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One-dimensional SDE's with Discontinuous, Unbounded Drift and Continuously Differentiable Solutions to the Stochastic Transport Equation

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Abstract

In this paper we develop a method for constructing strong solutions of one-dimensional SDE's where the drift may be discontinuous and unbounded. The driving noise is the Brownian Motion. In addition to existence and uniqueness of the strong solution, we show that the solution is Sobolev-differentiable in the initial condition and Malliavin differentiable. The method is based on Malliavin calculus using a similar technique as initiated in [11] and further developed in [10] and [12] where the authors consider bounded coefficients. This method is not based on a pathwise uniqueness argument. We will apply these results to the stochastic transport equation. More specifically, we obtain a continuously differentiable solution of the stochastic transport equation when the driving function is a step function.

Key words and phrases: irregular drift, stochastic flows, stochastic transport equations

MSC2010: 60H10, 60H15, 60H40.

1 Introduction

It is well known (see e.g. [7], page 303) that when b is sublinear, i.e.

$$|b(x)| \leq k_1 + k_2|x|, \tag{1}$$

the Stochastic Differential Equation (SDE)

$$\begin{cases} dX_t^x &= b(X_t^x)dt + dB_t \\ X_0^x &= x, \end{cases} \tag{2}$$

has a weak solution which is unique in the sense of probability law. In fact, this results holds for a possibly time inhomogenous coefficient b and in multipple dimensions. In this paper, we restrict our study to the one-dimensional autonomous equation. We will, however, study *strong* solutions of (2) and its regularity in the initial condition *and* its Malliavin differentiability.

SDE's with discontinuous coefficients have been an important area of study in stochastic analysis and other related fields of mathematics. In the theory of Ordinary Differential Equations (ODE's), the corresponding equation to (2) reads

$$\begin{cases} \frac{dX_t^x}{dt} &= b(X_t^x) \\ X_0^x &= x. \end{cases}$$

A solution to this equation may not be unique, and may not even exists when b is non-Lipschitz. However, adding a Brownian motion regularizes the equation. A breakthrough in the study of SDE's is the result by Zvonkin in [17]. Here, a global strong solution is constructed for b merely bounded and measurable. The technique is based on estimates of solutions of PDE's and the Yamada-Watanabe principle. Since the Yamada-Watanabe principle is an "indirect" technique, which relies on a purely measuretheoretical argument to obtain unique strong solutions of SDE's, the dependence of solutions on the initial condition is not so transparent.

Later, this subject has been studied by the authors in [10] and [12]. See also [4] where the authors use a different method to construct Sobolev-differentiable flow. The method is based on estimates on solutions to the backward Kolmogorov equation.

The method presented in this paper is based on Malliavin calculus coupled with probabilistic estimates on the weak derivative of the initial condition in the solution of the SDE (2).

In the study of stochastic (and deterministic) dynamical systems, the classical approach is to show that the flow 'inherits' the spatial regularity from the diffusion coefficient (see e.g. [8]). In this sense, the result presented in this paper (and in the papers mentioned above) is counter intuitive.

In this paper we will establish the existence of a Sobolev-differentiable stochastic flow

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (s,t,x) \mapsto \phi_{s,t}(x) \in \mathbb{R}$$

for the SDE

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x})du + B_t - B_s, \tag{3}$$

where b is assumed merely to satisfy (1), and the equation is driven by a standard one-dimensional Brownian motion B. The notion of Sobolev-differentiability will be in the sense that for a given $p > 1$ we have that $\phi_{s,t}(\cdot) \in L^2(\Omega; W^{1,p}(\mathbb{R}, e^{-x^4}dx))$ when $|t-s| \leq \delta$ where δ depends on k_2. Here, $W^{1,p}(\mathbb{R}, e^{-x^4}dx)$ denotes a weighted Sobolev-space. In addition, we shall show that $\phi_{s,t}(x) \in \mathbb{D}^{1,2}$ - the space of square-integrable Malliavin differentiable random variables.

The paper is organized as follows: In Section 2 we give the framework of the paper. The basic consepts of Gaussian white noise theory and Malliavin calculus is presented. In Section 3 we prove the existence of a solution to (3) with the above mentioned regularity. In Section 4.1 we use the technique in Section 3 to study 2 when b is a step-function. For such b it is shown that $x \mapsto X_t^x$ is in $C^\alpha(U)$ (Hölder space) for $U \subset \mathbb{R}$ open and bounded, and for all $\alpha < 1.5$. In particular, it is continuously differentiable. In Section 5 the results of Section 4.1 is applied to the stochastic transport equation:

$$\begin{aligned} d_t u(t,x) + b(x)\partial_x u(t,x)dt + \partial_x u(t,x) \circ dB_t = & \quad 0 \\ u(0,x) = & \quad u_0(x), \end{aligned} \tag{4}$$

where b is a step function and $u_0 \in C_b^1(\mathbb{R})$. Note that the corresponding deterministic transport equation is in general not well posed, even when b is continuous.

2 Framework

In this section we recall some facts from Gaussian white noise analysis and Malliavin calculus, which we aim at employing in Section 3 to construct strong solutions of SDE's. See [6, 14] for more information on white noise theory. As for Malliavin calculus the reader is referred to [13].

2.1 Basic Facts of Gaussian White Noise Theory

Throughout this paper we work with the white noise probability space

$$(\Omega, \mathcal{F}, \mu) = (\mathcal{S}'([0,T]), \mathcal{B}(\mathcal{S}'([0,T])), \mu),$$

where $\mathcal{S}'([0,T])$ is the dual space of $\mathcal{S}([0,T])$ - the Schwarz space on $[0,T]$. $\mathcal{B}(\mathcal{S}'([0,T]))$ is the Borel σ-algebra from the weak topology on $\mathcal{S}'([0,T])$ and μ is the probability measure such that

$$\int_{\mathcal{S}'([0,T])} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2([0,T])}}.$$

It can be verified that the process $B_t(\omega) = \langle \omega, 1_{[0,t]}(\cdot)$ obtained as a limit in $L^2(\Omega)$ is a Brownian motion.

The Wiener-Itô chaos decomposition (see e.g. [13]), gives that

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} I_n \left(\widehat{L}^2([0,T]^n) \right)$$

where $I_n : \widehat{L}^2([0,T]^n) \to L^2(\Omega)$ is the iterated Itô integral defined on $\widehat{L}^2([0,T]^n)$ - the subspace of $L^2([0,T]^n)$ consisting of symmetric functions.

Using the iterated Itô integral, one can lift the structure of the Gel'fand triple

$$\mathcal{S}([0,T]^n) \subset L^2([0,T]^n) \subset \mathcal{S}'([0,T]^n)$$

to construct a Gel'fand triple

$$(\mathcal{S}) \subset L^2(\Omega) \subset (\mathcal{S})^*.$$

We call $(\mathcal{S})$ the space of *Hida stochastic test functions* and $(\mathcal{S})^*$ the space of *Hida stochastic distributions.*

For an element $\Phi \in (\mathcal{S})^*$ we define its S-transform

$$(S\Phi)(\phi) = \langle \Phi, \mathcal{E}(\langle \cdot, \phi \rangle) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$, and

$$\mathcal{E}(\langle \cdot, \phi \rangle) = \exp\{\langle \cdot, \phi \rangle - \frac{1}{2}\|\phi\|^2_{L^2([0,T])}\}.$$

Here, $\phi \in \mathcal{S}_{\mathbb{C}}([0,T])$ - the complexification of $\mathcal{S}([0,T])$. It can be proved that if for $\Phi, \Psi \in (\mathcal{S})^*$ we have $S\Phi = S\Psi$, then $\Phi = \Psi$.

The Wick product of two elements $\Phi, \Psi \in (\mathcal{S})^*$ is defined as the unique element $\Phi \diamond \Psi \in (\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = (S\Phi)(\phi)(S\Psi)(\phi).$$

Finally, we mention a useful application of the white noise setting to the study of SDE's. This result was discovered in [9].

Proposition 2.1. *Suppose that the drift coefficient $b : \mathbb{R} \to \mathbb{R}$ in 2 is bounded and Lipschitz continuous. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu})$, $\tilde{B}$ be a copy of the quadruple $(\Omega, \mathcal{F}, \mu)$, B. Then the unique strong solution X_t^x allows for the explicit representation*

$$\varphi(X_t^x) = E_{\tilde{\mu}}[\varphi(x + \tilde{B}_t)\mathcal{E}_T^{\diamond}(b(\cdot + x))]$$

for all $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x + B_t) \in L^2(\Omega)$ for all $0 \leq t \leq T$. The object $\mathcal{E}_T^{\diamond}(b(\cdot + x))$ can be defined as the mapping

$$\mathcal{E}_T^{\diamond}(b(\cdot + x)) : \tilde{\Omega} \to (\mathcal{S})^*$$

such that composing with the S-transform (on the original tripel $(\Omega, \mathcal{F}, \mu)$), gives

$$S\mathcal{E}_T^{\diamond}(b(\cdot + x))(\phi) = \exp\left\{\int_0^T b(x + \tilde{B}_s) + \phi(s)d\tilde{B}_s - \frac{1}{2}\int_0^T (b(x + \tilde{B}_s) + \phi(s))^2 ds\right\}.$$

Here, $E_{\tilde{\mu}}$ denotes the Pettis integral of random variables $\Phi : \tilde{\Omega} \to (\mathcal{S})^*$ with respect to $\tilde{\mu}$.

2.2 Basic elements of Malliavin Calculus

In this Section we briefly elaborate a framework for Malliavin calculus.

We call a random variable smooth if it is on the form

$$F = f(\int_0^T h_1(s)dB_s, \dots, \int_0^T h_n(s)dB_s)$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $h_1, \dots h_n \in L^2([0,T])$. The *Malliavin* derivative of a smooth F is defined as the stochastic process

$$D_t F = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(\int_0^T h_1(s)dB_s, \dots, \int_0^T h_n(s)dB_s) h_i(t),$$

where $t \in [0,T]$. For a smooth random variable we may define the norm

$$\|F\|_{1,2}^2 = \|F\|_{L^2(\Omega)}^2 + \|D.F\|_{L^2(\Omega\times[0,T])}^2$$

and we denote by $\mathbb{D}^{1,2}$ the closure of the set of all smooth random variables with respect to $\|\cdot\|_{1,2}$. The Malliavin derivative operator D is then a closed linear operator from $\mathbb{D}^{1,2}$ to $L^2(\Omega \times [0,T])$. We shall say that a random variable is Malliavin differentiable if it is in $\mathbb{D}^{1,2}$.

3 Main Results

In this section we will study the SDE

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x})du + B_t - B_s$$

where the drift coefficient $b : \mathbb{R} \to \mathbb{R}$ is merely measurable and sublinear:

$$|b(x)| \le k_1 + k_2|x|.$$

It is known that the above SDE has a unique strong solution in the case of $k_2 = 0$, and a weak solution when $k_2 > 0$, unique in the sense of probability law.

Here we will establish the existence of a Sobolev differentiable flow of homeomorphisms for the SDE.

Definition 3.1. *A map $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (s,t,x,\omega) \longmapsto \phi_{s,t}(x,\omega) \in \mathbb{R}$ is a stochastic flow of homeomorphisms for the SDE (3) if there exists a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that for all $\omega \in \Omega^*$, the following statements are true:*
(i) For any $x \in \mathbb{R}$, the process $\phi_{s,t}(x,\omega), s,t \in \mathbb{R}$, is a strong global solution to the SDE (3).
(ii) $\phi_{s,t}(x,\omega)$ is continuous in $(s,t,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$.
(iii) $\phi_{s,t}(\cdot,\omega) = \phi_{u,t}(\cdot,\omega) \circ \phi_{s,u}(\cdot,\omega)$ for all $s,u,t \in \mathbb{R}$.
(iv) $\phi_{s,s}(x,\omega) = x$ for all $x \in \mathbb{R}$ and $s \in \mathbb{R}$.
(v) $\phi_{s,t}(\cdot,\omega) : \mathbb{R} \to \mathbb{R}$ are homeomorphisms for all $s,t \in \mathbb{R}$.

A stochastic flow $\phi_{s,t}(\cdot,\omega)$ of homeomorphisms is said to be *Sobolev-differentiable* if for all $s,t\in\mathbb{R}$, the maps $\phi_{s,t}(\cdot,\omega)$ and $\phi_{s,t}^{-1}(\cdot,\omega)$ are Sobolev-differentiable in the sense described below.

In order to prove the existence of a Sobolev differentiable flow for the SDE (3), we need to introduce a suitable class of weighted Sobolev spaces. Let $L^p(\mathbb{R},e^{-x^4}dx)$ denote the space of all Borel measurable functions $u:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$\int_{\mathbb{R}}|u(x)|^p\,e^{-x^4}dx<\infty. \tag{5}$$

Furthermore, denote by $W^{1,p}(\mathbb{R},e^{-x^4}dx)$ the linear space of functions $u\in L^p(\mathbb{R},e^{-x^4}dx)$ with a weak derivatives $Du\in L^p(\mathbb{R},e^{-x^4}dx)$. We equip this space with the complete norm

$$\|u\|_{1,p}:=\|u\|_{L^p(\mathbb{R},e^{-x^4}dx)}+\|Du\|_{L^p(\mathbb{R},e^{-x^4}dx)}\,. \tag{6}$$

We will show that the strong solution $X_t^{s,\cdot}$ of the SDE (3) is in $L^2(\Omega,L^p(\mathbb{R},e^{-x^4}dx))$ when $p>1$. In fact, the SDE (3) implies the following estimate:

$$\begin{aligned}|X_t^{s,x}|^p&\le c_p(|x|^p+k_1^p|t-s|^p+k_2^p\int_s^t|X_u^{s,x}|^pdu+|B_t-B_s|^p)\\&\le c_p(|x|^p+k_1^p|t-s|^p+|B_t-B_s|^p)e^{k_2^p|t-s|},\end{aligned}$$

where the last inequality is due to Gronwall's lemma.

In particular, for fixed x we have $X_t^{s,x}\in L^2(\Omega)$. From Proposition 3.10 page 304 in [7] we get that a solution, if it exists, must be unique in law.

On the other hand, it is easy to see that solutions $X_t^{s,\cdot}$ are in general not in $L^p(\mathbb{R},dx)$ with respect to the Lebesgue measure dx on $\mathbb{R}$: Just consider the special trivial case $b\equiv 0$. This implies that solutions of the SDE (3) (if they exist) may not belong to the Sobolev space $W^{1,p}(\mathbb{R},dx),p>1$. However, we will show that such solutions do indeed belong to the weighted Sobolev spaces $W^{1,p}(\mathbb{R},e^{-x^4}dx)$ for $p\ge 1$.

We now state our main result in this section which gives the existence of a Sobolev differentiable stochastic flow for the SDE (3).

Theorem 3.2. *There exists a stochastic flow $\phi_{s,t}(x)$ of the SDE (3). Moreover, the flow is differentiable on small time intervals in the sense that given $p>1$ there exists a $\delta>0$ such that*

$$\phi_{s,t}(\cdot)\ \text{ and }\ \phi_{s,t}^{-1}(\cdot)\in L^2(\Omega;W^{1,p}(\mathbb{R},e^{-x^4}dx)),$$

whenever $|t-s|\le\delta$.

Remark 3.3. *Note from the proof of Lemma 3.5 that the size of $\delta > 0$ depends only on k_2. In particular, if $k_2 = 0$, i.e. the function is bounded, we have that $x \mapsto \phi_{s,t}(x)$ is weakly differentiable for every $t, s \in \mathbb{R}$.*

We will prove this theorem through a sequence of lemmas and propositions. We begin by stating the main proposition:

Proposition 3.4. *Let $b : \mathbb{R} \to \mathbb{R}$ be measurable and sublinear. Let U be an open and bounded subset of $\mathbb{R}$ and $p \geq 1$. Then there exists a $T > 0$ such that there exists a solution X_t^x to the SDE (2) on $[0, T]$. Morover, we have*

$$X_t^{\cdot} \in L^2(\Omega; W^{1,p}(U)),$$

and for each $t \in [0, T]$ and $x \in \mathbb{R}$, $X_t^x \in \mathbb{D}^{1,2}$.

We shall prove Proposition 3.4 in a smiliar manner as in [12]. That is, we assume first that b is smooth and has compact support. It is then possible to bound the Malliavin derivative, $D.X_t^x$, and the spatial derivative, $\frac{d}{dx}X_t^x$, independently of the size of b'. In fact, we will use a bound depending only on k_1 and k_2 from (1).

Then, assuming b to be merely sublinear, we pick a sequence $\{b_n\}_{n\geq 1}$ of smooth functions with compact support such that $b_n(x) \to b(x)$ Lebesgue almost everywhere, and such that

$$\sup_{n\geq 1} |b_n(x)| \leq k_1 + k_2|x|.$$

We denote by $X_t^{n,x}$ the corresponding sequence of solutions of (2) when b is replaced by b_n. Using the a priori estimates in Lemma 3.5 in connection with a compactness criterium based on Malliavin Calculus we can extract a converging subsequence in the strong topology of $L^2(\Omega)$ and verify that this limit is in fact the solution to (2). Moreover, since $X_t^{n,\cdot}$ is also bounded in $L^2(\Omega; W^{1,p}(U))$ we use a weak compactness argument to show that the limit is also in $L^2(\Omega; W^{1,p}(U))$.

We now turn to proving the a priori estimates. Note that when b is a compactly supported smooth function, the corresponding solution of the SDE (2) is both Malliavin differentiable and continuously differentiable with respect to x. Moreover, these derivatives can be expressed through the following linear ODE's, respectively (see [13] and [8], respectively)

$$D_s X_t^x = 1 + \int_s^t b'(X_u^x) D_s X_u^x du, \text{ for } s < t \tag{7}$$

and

$$\frac{d}{dx} X_t^x = 1 + \int_0^t b'(X_u^x) \frac{d}{dx} X_u^x du. \tag{8}$$

Lemma 3.5. *Let $b \in C^1$ have compact support. We may choose $T > 0$ such that there exists constants $C = C(k_1, k_2, T)$ and c (independent of k_1, k_2 and T) such that for every $t, s \in [0, T]$, $s < t$ we have*

$$E[(D_s X_t^x)^2] \leq e^{cTk_2^2 x^2} C(t-s)^{-1/4} s^{-1/4} \tag{9}$$

and

$$E[(D_{s_1} X_t^x - D_{s_2} X_t^x)^2] \leq e^{cTk_2^2 x^2} C(s_2 - s_1) \left((t-s_1)^{-1/8} s_1^{-1/8} + (t-s_2)^{-1/8} s_2^{-1/8} \right). \tag{10}$$

Proof. We note first that the linear ODE (7) is uniquely solved by

$$D_s X_t^x = \exp\{\int_s^t b'(X_u^x) du\}.$$

Using the Girsanov theorem we get

$$E[(D_s X_t^x)^2] = E[\exp\{2 \int_s^t b'(x + B_u) du\} \mathcal{E}(\int_0^1 b(x + B_u) dB_u)].$$

By Itô's formula, with $\tilde{b}(z) := \int_{-\infty}^z b(y) dy$ we have

$$\tilde{b}(x + B_t) = \tilde{b}(x + B_s) + \int_s^t b(x + B_u) dB_u + \frac{1}{2} \int_s^t b'(x + B_u) du$$

so that

$$E[(D_s X_t)^2] = E[\exp\{4(\tilde{b}(x+B_t) - \tilde{b}(x+B_s) - \int_s^t b(x+B_u) dB_u)\} \mathcal{E}(\int_0^T b(x+B_u) dB_u)]$$

$$\leq \| \exp\{4(\tilde{b}(x+B_t) - \tilde{b}(x+B_s))\} \|_{L^2(\Omega)} \| \exp\{-4 \int_s^t b(x+B_u) dB_u)\} \mathcal{E}(\int_s^t b(x+B_u) dB_u) \|_{L^2(\Omega)}$$

by Hölder's inequality. For the first term, by (1)

$$\begin{aligned} |\tilde{b}(x + B_t) - \tilde{b}(x + B_s)| &= | \int_0^1 b(x + B_s + \theta(B_t - B_s)) d\theta || B_t - B_s| \\ &\leq \int_0^1 k_1 + k_2 |x + B_s + \theta(B_t - B_s)| d\theta |B_t - B_s| \\ &\leq k_1 + k_2 |x + B_s| |B_t - B_s| + \frac{k_2}{2}(B_t - B_s)^2 \\ &\leq k_1 + \frac{k_2}{4} x^2 + \frac{k_2}{4} B_s^2 + k_2 (B_t - B_s)^2, \end{aligned}$$

so that

$$E[\exp\{8(\tilde{b}(x+B_t)-\tilde{b}(x+B_s))\}] \leq e^{8k_1+2k_2x^2} E[\exp\{2k_2B_s^2+8k_2(B_t-B_s)^2\}]$$

$$= e^{8k_1+2k_2x^2} E[\exp\{2k_2B_s^2\}] E[\exp\{8k_2(B_t-B_s)^2\}]$$

$$= e^{8k_1+2k_2x^2}(2\pi)^{-1}((t-s)s)^{-1/2} \int_{\mathbb{R}} \exp\{4k_2z^2+\frac{-z^2}{2s}\}dz \int_{\mathbb{R}} \exp\{8k_2z^2\frac{-z^2}{2(t-s)}\}dz,$$

where we have used independence of the increments of the Brownian motion. Both integrals are finite for small T. This gives that

$$\|\exp\{4(\tilde{b}(x+B_t)-\tilde{b}(x+B_s))\}\|_{L^2(\Omega)} \leq (t-s)^{-1/4}s^{-1/4}e^{4k_1+k_2x^2}\sqrt{c_1(k_2)},$$

where

$$c_1(k_2) := (2\pi)^{-1} \int_{\mathbb{R}} \exp\{z^2(4k_2-\frac{1}{2T})\}dz \int_{\mathbb{R}} \exp\{z^2(8k_2-\frac{1}{2T})\}dz.$$

For the second term consider

$$E[\exp\{-8\int_s^t b(x+B_u)dB_u\}\mathcal{E}(\int_s^t b(x+B_u)dB_u)^2]$$

$$= E[\exp\{-6\int_s^t b(x+B_u)dB_u - \int_s^t b^2(x+B_u)du\}]$$

$$= E[\exp\{-6\int_s^t b(x+B_u)dB_u - \alpha\int_s^t b^2(x+B_u)du\}\exp\{(\alpha-1)\int_s^t b^2(x+B_u)du\}]$$

$$\leq \|\exp\{-6\int_s^t b(x+B_u)dB_u - \alpha\int_s^t b^2(x+B_u)du\|_{L^2(\Omega)}$$

$$\times\|\exp\{(\alpha-1)\int_s^t b^2(x+B_u)du\}\|_{L^2(\Omega)}.$$

If we now choose α such that $\frac{1}{2}(-12b(x+B_u))^2 = 2\alpha b^2(x+B_u)$, that is $\alpha = 36$, the process $\exp\{-12\int_s^t b(x+B_u)dB_u - 36\int_s^t b^2(x+B_u)du = \mathcal{E}(\int_s^t(-12b(x+B_u))dB_u)$ is a martingale, hence has expectation equal to 1. Using (1), we get that the second term is bounded by

$$E[\exp\{70\int_s^t b^2(x+B_u)du\}] \leq E[\exp\{70\int_s^t (k_1+k_2|x+B_u|)^2du\}]$$

$$\leq E[\exp\{70(t-s)(k_1+k_2\max_{0\leq u\leq t}|x+B_u|)^2\}].$$

Define $Y_u = \exp\{35(t-s)(k_1 + k_2|x + B_u|)^2\}$ which is readily seen to be a submartingale. By Doob's Maximal inequality we get the following bound

$$
\begin{aligned}
E[\exp\{70(t-s)(k_1 + k_2 x + B_t^*)^2\}] &= E[\sup_{u\le t} Y_u^2] \\
&\le 4E[Y_t^2] = 4E[\exp\{70(t-s)(k_1 + k_2|x + B_t|)^2\}] \\
&\le 4(2\pi t)^{-1/2}\exp\{170t(k_1^2 + 2k_2 x^2)\} \\
&\times \int_{\mathbb{R}} \exp\{(140tk_2^2 - \frac{1}{2t})z^2\}dz.
\end{aligned}
$$

The latter integral is finite for small T. This proves (9).

For the second estimate, assume $s_1 \le s_2$ we write

$$
\begin{aligned}
D_{s_1}X_t^x - D_{s_2}X_t^x &= \exp\{\int_{s_1}^t b'(X_u^x)du\} - \exp\{\int_{s_2}^t b'(X_u^x)du\} \\
&\le \left|\exp\{\int_{s_1}^t b'(X_u^x)du\} + \exp\{\int_{s_2}^t b'(X_u^x)du\}\right| \, |\int_{s_1}^{s_2} b'(X_u^x)du|,
\end{aligned}
$$

where we have used the inequality $|e^y - e^z| \le |e^y + e^z||y-z|$. Using Girsanov's theorem we get

$$
\begin{aligned}
&E[(D_{s_1}X_t^x - D_{s_2}X_t^x)^2] \\
&\le E[(\int_{s_1}^{s_2} b'(x+B_u)du)^2 \times \left(\exp\{\int_{s_1}^t b'(X_u^x)du\} + \exp\{\int_{s_2}^t b'(X_u^x)du\}\right)^2 \mathcal{E}(\int_0^T b(x+B_u)dB_u)] \\
&\le \|(\int_{s_1}^{s_2} b'(x+B_u)du)^2\|_{L^2(\Omega)} 2\left(\|\exp\{\int_{s_1}^t 2b'(X_u^x)du\}\mathcal{E}(\int_0^T b(x+B_u)dB_u)\|_{L^2(\Omega)} + \right. \\
&\qquad \left. \|\exp\{\int_{s_2}^t 2b'(X_u^x)du\}\mathcal{E}(\int_0^T b(x+B_u)dB_u)\|_{L^2(\Omega)}\right).
\end{aligned}
$$

For the first term rewrite

$$
\begin{aligned}
(\int_{s_1}^{s_2} b'(x+B_u)du)^4 \le& 2^3\left(\int_0^1 b(x + B_{s_1} + \theta(B_{s_2} - B_{s_1}))d\theta(B_{s_2} - B_{s_1})\right)^4 \\
&+ 2^3\left(\int_{s_1}^{s_2} b(x+B_u)dB_u\right)^4 \\
\le& 2^3\left(k_1 + k_2|x + B_{s_2}| + \frac{k_2}{2}|B_{s_2} - B_{s_1}|\right)^4 (B_{s_2} - B_{s_1})^4 \\
&+ 2^3\left(\int_{s_1}^{s_2} b(x+B_u)dB_u\right)^4.
\end{aligned}
$$

We can estimate

$$E[(\int_{s_1}^{s_2} b(x+B_u)dB_u)^4] \leq 36(s_2-s_1)\int_{s_1}^{s_2} E[(b(x+B_u))^4]du$$
$$\leq 36(s_2-s_1)^2 \sup_{s_1\leq u\leq s_2} E[(b(x+B_u))^4],$$

which is finite since b satisfies (1).

Similarly as before, we may estimate

$$\|\exp\{\int_s^t 2b'(X_u^x)du\}\mathcal{E}(\int_0^T b(x+B_u)dB_u)\|_{L^2(\Omega)} \leq e^{cTk_2^2x^2}C(t-s)^{-1/8}s^{-1/8}.$$

This proves (10).

□

We see that equation (8) is the same equation as (7) when we put $s=0$. Using this fact in connection with a similar proof as above, replacing the Malliavin derivative D_s by $\frac{d}{dx}$, we immediately arrive at the following result:

Proposition 3.6. *Let $b \in C^1$ have compact support, and let $p \geq 1$. We may choose $T > 0$ such that there exists constants $C = C(k_1, k_2, T, p)$ and c (independent of k_1, k_2, p and T) such that*

$$E[|\frac{d}{dx}X_t^x|^p] \leq e^{cpTk_2^2x^2}Ct^{-1/2}. \tag{11}$$

Using Lemma 3.5 together with Corollary 6.3 we immediately obtain the following Corollary:

Corollary 3.7. *Let $b_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$ be a sequence of continuously differentiable bounded functions that satisfies (1) uniformly in n, i.e.*

$$\sup_{n\geq 1}|b_n(x)| \leq k_1 + k_2|x|.$$

Denote by $X_t^{n,x}$ the corresponding sequence of strong solutions. Then $\{X_t^{n,x}\}_{n\geq 1}$ is relatively compact in $L^2(\Omega)$.

We are now ready to prove that the SDE (2) has a strong solution.

Proposition 3.8. *Retain the above assumptions and notation. Define*

$$X_t^x := E_{\hat{\mu}}[(x+\widehat{B}_t)\mathcal{E}_T^{\diamond}(b(\cdot+x))]$$

which is a well defined random variable in $L^2(\Omega)$. Assume that $b_n(y) \to b(y)$ Lebesgue almost every $y \in \mathbb{R}$. Then there exists a $T > 0$ and a subsequence $X_t^{n(k),x}$ which converges in $L^2(\Omega)$ to X_t^x for all $t \in [0,T]$. Moreover, X_t^x is the unique solution to (2) and it is Malliavin differentiable, that is $X_t^x \in \mathbb{D}^{1,2}$.

Proof. By Corollary 3.7 we know that there exists a subsequence, still denoted $X_t^{n,x}$ for simplicity, converging in $L^2(\Omega)$. The above definition of X_t^x is a well defined object in $(\mathcal{S})^*$ for small T (see [11], Lemma 11). Taking the S-transform we get (see [11], Lemma 12)

$$|S(X_t^{n,x})(\phi) - S(X_t^x)(\phi)| \leq C(E[J_n(x)])^{1/2} \exp\{34 \int_0^T |\phi(s)|^2 ds\}$$

where

$$\begin{aligned} J_n(x) := {} & 2 \int_0^T (b_n(x+B_u) - b(x+B_u))^2 \, du \\ & + \left(\int_0^T \left| (b_n(x+B_u))^2 - (b(x+B_u))^2 \right| du \right)^2 . \end{aligned}$$

By the uniform sublinearity we may invoke dominated convergence to conclude that $E[J_n(x)] \to 0$ as $n \to \infty$, so that $X_t^{n,x} \to X_t^x$ in $(\mathcal{S})^*$. It follows that this convergence is actually in $L^2(\Omega)$ by uniqueness of the limits.

We now claim that for any function φ such that $\varphi(x + B_t) \in L^2(\Omega)$ we have

$$\varphi(X_t^x) = E_{\tilde{\mu}}[\varphi(x + B_t)\mathcal{E}^\diamond(b(\cdot + x))]. \tag{12}$$

To see this, assume first that $\varphi \in C_b^1(\mathbb{R})$. We know from Proposition 2.1 that for every n we have

$$\varphi(X_t^{n,x}) = E_{\tilde{\mu}}[\varphi(x + B_t)\mathcal{E}^\diamond(b_n(\cdot + x))].$$

We have that $\varphi(X_t^{n,x}) \to \varphi(X_t^x)$ in $L^2(\Omega)$ since

$$E[|\varphi(X_t^{n,x}) - \varphi(X_t^x)|^2] \leq \|\varphi'\|_\infty^2 E[|X_t^{n,x} - X_t^x|^2].$$

On the other hand we get that $\varphi(X_t^{n,x}) \to E_{\tilde{\mu}}[\varphi(x+B_t)\mathcal{E}^\diamond(b(\cdot+x))]$ in $(\mathcal{S})^*$ as long as $E[J_n(x)] \to 0$ by a similar argument as in [11], Lemma 12. This proves (12) for $\varphi \in C_b^1(\mathbb{R})$. The general case follows by approximation in connection with the monotone class theorem.

To verify that X_t^x indeed solves (2), notice that $\widetilde{B_t}$ is a weak solution to (2) if the drift is replaced by $b(\cdot) + \phi(s)$ with respect to the measure

$$d\mu^* = \mathcal{E}\left(\int_0^T b(\widetilde{B}_u) + \phi(u) d\widetilde{B}_u \right) d\tilde{\mu}.$$

Taking the S-transform we get

$$\begin{aligned}
S(X_t^x)(\phi) &= E_{\tilde{\mu}}[\widetilde{B}_t \mathcal{E}\left(\int_0^T b(\widetilde{B}_s) + \phi(s) d\widetilde{B}_s\right)] \\
&= E_{\mu^*}[\widetilde{B}_t] \\
&= E_{\mu^*}[\int_0^t b(\widetilde{B}_s) + \phi(s) ds] \\
&= \int_0^t E_{\tilde{\mu}}[b(\widetilde{B}_s)\mathcal{E}\left(\int_0^T b(\widetilde{B}_u) + \phi(u) d\widetilde{B}_u\right)] ds + S(B_t)(\phi).
\end{aligned}$$

By (12) we get that

$$S(X_t^x)(\phi) = S(\int_0^t b(X_s^x) ds)(\phi) + S(B_t)(\phi).$$

Since S is injective, this proves that the $\mathcal{F}_t$-adapted X_t^x solves the equation.

Since $\sup_{n\geq 1} \|D.X_t^{n,x}\|_{L^2(\Omega\times[0,T])} < \infty$ it follows that $X_t^x \in \mathbb{D}^{1,2}$.

To see that X_t^x is the unique solution to (2) we first note that for T small enough, the Novikov condition is satisfied with respect to $b(X_{cdot}^x)$, since

$$E[\exp\{\frac{1}{2}\int_0^T |b(X_s^x)|^2 ds\}] = E[\exp\{\frac{1}{2}\int_0^T |b(x+B_s)|^2 ds\}\mathcal{E}(\int_0^T b(x+B_s) dB_s)]$$

$$\leq \| \exp\{\frac{1}{2}\int_0^T |b(x+B_s)|^2 ds\}\|_{L^2(\Omega)} \|\mathcal{E}(\int_0^T b(x+B_s) dB_s)\|_{L^2(\Omega)}.$$

Since the solution is unique in law, the Novikov condition is then automatically satisfied for any other strong solution. Then the proof of Proposition 2.1 (see e.g. [9]) shows that any other solution necessarily takes the form

$$E_{\hat{\mu}}[(x + \widehat{B}_t)\mathcal{E}_T^{\diamond}(b(\cdot + x))]$$

□

We are now ready to prove Proposition 3.4.

Proof of 3.4. Existence, uniqueness and Malliavin differentiability of X_t^x is contained in Proposition 3.8. It remains to show that $X_t^{\cdot} \in L^2(\Omega; W^{1,p}(U))$. To this end, we observe that given $p \geq 1$, there exists a $T > 0$ such that for any $\varphi \in C_0^\infty(U)$ the sequence

$$\langle X_t^n, \varphi\rangle := \int_U X_t^{n,x}\varphi(x) dx$$

is relatively compact for $t \in [0,T]$. To see this we use the compactness criterion of Corollary 6.3. Note that since the Malliavin derivative is a closed linear operator we have

$$E[(D_s\langle X_t^n,\varphi\rangle)^2] = E[(\int_U D_s X_t^{n,x}\varphi(x)dx)^2] \leq \|\varphi\|^2_{L^2(U)} leb(U) \sup_{x\in U} E[(D_s X_t^{n,x})^2]$$

and similary

$$E[(D_{s_1}\langle X_t^n,\varphi\rangle - D_{s_2}\langle X_t^n,\varphi\rangle)^2] \leq \|\varphi\|^2_{L^2(U)}\lambda(U)\sup_{x\in U} E[(D_{s_1}X_t^{n,x} - D_{s_2}X_t^{n,x})^2],$$

which shows that $\langle X_t^n,\varphi\rangle$ is relatively compact. Denote by $Y_t(\varphi)$ its limit after taking an (if necessary) subsequence.

Taking the S-transform of $\langle X_t^n,\varphi\rangle$ and $\langle X_t,\varphi\rangle$ we see that for any $\phi \in \mathcal{S}_{\mathbb{C}}([0,T])$

$$\begin{aligned} |S(\langle X_t^n,\varphi\rangle)(\phi) - S(\langle X_t,\varphi\rangle)(\phi)|^2 &= |\langle S(X_t^n - X_t)(\phi),\varphi\rangle|^2 \\ &\leq \|\varphi\|^2_{L^2(\mathbb{R})}\int_U |S(X_t^{n,x} - X_t^x)(\phi)|^2 dx \\ &\leq \|\varphi\|^2_{L^2(\mathbb{R})}\int_U CE[J_n(x)]\exp(68\int_0^T \|\phi(s)\|^2 ds)dx, \end{aligned}$$

where C is a constant and $J_n(x)$ as in Proposition 3.8. Since $\{b_n\}$ is uniformly sublinear, using dominated convergence, we get that

$$\langle X_t^n,\varphi\rangle \to \langle X_t,\varphi\rangle$$

in $(\mathcal{S})^*$, and thus in particular weakly in $L^2(\Omega)$. By uniqueness of the limits we can conclude that

$$Y(\varphi) = \langle X_t,\varphi\rangle \;\; \mu\text{-a.s.},$$

thus proving the assertion.

Note that there exists a subsequence $n(k)$ such that $\langle X_t^{n(k)},\varphi\rangle$ converges for every φ, that is, $n(k)$ is independent of φ. To see this, let $x=0$ and choose $n(k)$ such that

$$X_t^{n(k),0} \to X_t^0$$

in $L^2(\Omega)$. If there exists $\varphi \in C_0^\infty(\mathbb{R})$ and $\epsilon > 0$ such that $\|\langle X_t^{n(k)},\varphi\rangle - \langle X_t,\varphi\rangle\| \geq \epsilon$ we may by the above extract a further subsequence $\langle X_t^{n(k(j))},\varphi\rangle$ converging to $\langle X_t,\varphi\rangle$, which gives a contradiction. From now we denote this subsequence by n for simplicity.

We now proceed to prove that $(x \mapsto X_t^x) \in L^2(\Omega; W^{1,p}(U))$: Because of Lemma 3.5 we get that $(x \mapsto X_t^{n,x})$ is bounded in $L^2(\Omega; W^{1,p}(U))$, thus relatively compact in the weak topology. Then there exists a subsequence

$n(k)$ such that $X_t^{n(k),\cdot}$ converges weakly to an $Y \in L^2(\Omega; W^{1,p}(U))$. Then for all $A \in \mathcal{F}$ and $\varphi \in C_0^\infty$ we have

$$E[1_A \langle X_t, \varphi' \rangle] = \lim_{k\to\infty} E[1_A \langle X_t^{n(k)}, \varphi' \rangle]$$

$$= \lim_{k\to\infty} -E[1_A \langle \frac{d}{dx} X_t^{n(k)}, \varphi \rangle] = -E[1_A \langle Y, \varphi \rangle].$$

Hence we have

$$\langle X_t, \varphi' \rangle = -\langle Y, \varphi \rangle \ \ \mu\text{-a.s.} \tag{13}$$

Finally, we need to show that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that $X_t^\cdot$ has a weak derivative on this subset. To this end choose a sequence $\{\varphi_n\}$ in $C^\infty(\mathbb{R})$ dense in $W_0^{1,p}(U)$. Choose a measurable subset Ω_n of Ω with full measure such that (13) holds on Ω_n with φ replaced by φ_n. Then $\Omega_0 := \cap_{n\geq 1} \Omega_n$ satisfies the desired property.

$\square$

Remark 3.9. *By a similar argument as in the above proof, one can show that there exists a subsequence $n(k)$ such that $X_t^{n(k),x} \to X_t^x$ in $L^2(\Omega)$ for all t and x, i.e. the choice of subsequence is independent of t and x. From now on we shall always use this subsequence, for simplicity denoted by n.*

Lemma 3.10. *For all $p \in (1,\infty)$ there exists a $T > 0$ such that we have*

$$X_t^\cdot \in L^2(\Omega, W^{1,p}(\mathbb{R}, e^{-x^4}dx))$$

for every $t \in [0,T]$.

Proof. It suffices to show that $E[(\int_{\mathbb{R}} |\frac{d}{dx} X_t^x|^p e^{-x^4} dx)^{2/p}] < \infty$. To this end, let $X_t^{n,x}$ denote the sequence approximating X_t^x as in the previous lemma. Assume first that $p \geq 2$. Then by Hölder's inequality w.r.t. μ we have

$$E[(\int_{\mathbb{R}} |\frac{d}{dx} X_t^{n,x}|^p e^{-x^4} dx)^{2/p}]$$

$$\leq \left(E[\int_{\mathbb{R}} |\frac{d}{dx} X_t^{n,x}|^p e^{-x^4} dx \right)^{2/p}],$$

which is finite from Fubini's theorem in connection with the bound in 3.5. For $1 < p \leq 2$, by Hölder's inequality w.r.t. $e^{-x^4}dx$ we have

$$E[(\int_{\mathbb{R}} |\frac{d}{dx} X_t^{n,x}|^p e^{-x^4} dx)^{2/p}] \leq (\int_{\mathbb{R}} w(x) dx)^{(2-p)/p} \int_{\mathbb{R}} E[|\frac{d}{dx} X_t^{n,x}|^2] e^{-x^4} dx.$$

In both cases we can find a subsequence converging to an element $Y \in L^2(\Omega, L^p(\mathbb{R}, e^{-x^4}dx))$ in the weak topology, in particular for every $A \in \mathcal{F}$ and $f \in L^q(\mathbb{R}, e^{-x^4}dx)$ (q is the Hölder conjugate of p) we have

$$\lim_{k\to\infty} E[1_A \int_{\mathbb{R}} \frac{d}{dx} X_t^{n(k),x} f(x) e^{-x^4} dx] = E[1_A \int_{\mathbb{R}} Y(x) f(x) e^{-x^4} dx].$$

If we let $f(x) = e^{x^4}\varphi(x)$ for $\varphi \in C_0^\infty(\mathbb{R}))$ we see from the previous theorem that Y must coincide with the weak derivative of X_t^x. This proves the claim. $\square$

In the sequel let us denote by

$$\int_{\mathbb{R}} f(y) dL_t^y(X_\cdot^x) \tag{14}$$

the integral of a measurable function $f : \mathbb{R} \to \mathbb{R}$ with respect to the local time of X_t^x in space. For more information about local time spatial integration, the reader is referred to [2] and [15].

Proposition 3.11. *The spatial and Malliavin derivatives of the solution X_t^x to (2) have the following explicit representations, respectively,*

$$\frac{d}{dx} X_t^x = \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^x)\} \tag{15}$$

$$= \exp\{2\int_0^1 b(\theta X_t^x + (1-\theta)x) d\theta (X_t^x - x) - 2\int_0^t b(X_u^x) dX_u^x\} \tag{16}$$

which holds $\lambda \times \mu$ almost everywhere, and for a fixed $x \in \mathbb{R}$ we have

$$D_s X_t^x = \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^x) + \int_{\mathbb{R}} b(y) dL_s^y(X_\cdot^x)\} \tag{17}$$

$$= \exp\{2\int_0^1 b(\theta X_t^x + (1-\theta)X_s^x) d\theta (X_t^x - X_s^x) - 2\int_s^t b(X_u^x) dX_u^x\} \tag{18}$$

μ-almost surely.

Proof. We will prove that the following convergence

$$\frac{d}{dx} X_t^{n,\cdot} \to \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^\cdot)\}$$

holds weakly in $L^2(U \times \Omega)$ for any $U \subset \mathbb{R}$ open and bounded. This will prove (15). To see (16), we refer to [15].

To this end we will use the fact that the set of functions $\{\varphi \otimes \exp\{\int_0^t h(s) dB_s\}\}$ is total in $L^2(U \times \Omega)$ when φ ranges through $C_0^\infty(U)$ and h ranges through the step functions defined on $[0, T]$. We have by the Girsanov's theorem

$$\left| \langle \varphi \otimes \exp\{\int_0^t h(s) dB_s\}, \frac{d}{dx} X_t^{n,\cdot} - \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^\cdot)\} \rangle_{L^2(U\times\Omega)} \right|$$

$$= \left| \int_{\mathbb{R}} \varphi(x) E[\exp\{\int_0^t h(s) dX_s^{n,x}\} \exp\{-\int_{\mathbb{R}} b_n(y) dL_t^y(x + B_\cdot)\} \mathcal{E}(\int_0^t b_n(x + B_u) dB_u)] dx \right.$$

$$
-\int_{\mathbb{R}} \varphi(x) E[\exp\{\int_0^t h(s) dX_s^x\} \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(x+B_\cdot)\} \mathcal{E}(\int_0^t b(x+B_u) dB_u)] dx \Big|
$$

$$
\begin{aligned}
&\leq \Big| \int_{\mathbb{R}} \varphi(x) E[\Big(\exp\{\int_0^t h(s) dX_s^{n,x}\} - \exp\{\int_0^t h(s) dX_s^x\} \Big) \\
&\quad \times \exp\{\int_0^t b_n'(x+B_s) ds\} \mathcal{E}(\int_0^t b_n(x+B_u) dB_u)] dx \Big| \\
&+ \Big| \int_{\mathbb{R}} \varphi(x) E[\exp\{\int_0^t h(s) dX_s^x\} \\
&\quad \times \Big(\exp\{\int_0^t b_n'(x+B_s) ds\} - \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(x+B_\cdot)\Big) \mathcal{E}(\int_0^t b_n(x+B_u) dB_u)] dx \Big| \\
&+ \Big| \int_{\mathbb{R}} \varphi(x) E[\exp\{\int_0^t h(s) dX_s^x\} \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(x+B_\cdot)\} \\
&\quad \times \Big(\mathcal{E}(\int_0^t b_n(x+B_u) dB_u) - \mathcal{E}(\int_0^t b(x+B_u) dB_u) \Big)] dx \Big| \\
&\quad =: i)_n + ii)_n + iii)_n.
\end{aligned}
$$

For the first term, since $\{\exp\{\int_0^t b_n'(x+B_s) ds\} \mathcal{E}(\int_0^t b_n(x+B_u) dB_u)\}_{n\geq 1}$ is bounded in $L^2(\Omega)$ provided T is small enough we have

$$
\begin{aligned}
i)_n \leq \int_U |\varphi(x)| \| \exp\{\int_0^t h(s) dX_s^{n,x}\} - \exp\{\int_0^t h(s) dX_s^x\} \|_{L^2(\Omega)} \times \\
\| \exp\{\int_0^t b_n(x+B_s) ds\} \mathcal{E}(\int_0^t b_n(x+B_u) dB_u) \|_{L^2(\Omega)} dx
\end{aligned}
$$

We know that $X_t^{n,x} \to X_t^x$ for all t and x (see Remark 3.9) in $L^2(\Omega)$. In particular, there exists a subsequence (still denoted n for simplicity) converging μ almost surely. Since h is a step function we get $\int_0^t h(s) dX_s^{n,x} \to \int_0^t h(s) dX_s^x$ μ almost surely. By dominated convergence we have

$$
\lim_{n\to\infty} i)_n = 0.
$$

For the second term we use (see [15]) the following equality

$$
\begin{aligned}
\int_0^t b_n'(x+B_s) ds = -\int_{\mathbb{R}} b_n(y) dL_t^y(x+B_\cdot) \\
= 2\int_0^1 b_n(x+\theta B_t) d\theta B_t - 2\int_0^t b_n(x+B_s) dB_s
\end{aligned}
$$

and

$$
\int_{\mathbb{R}} b(y) dL_t^y(x+B_\cdot) = -2\int_0^1 b(x+\theta B_t) d\theta B_t + 2\int_0^t b(x+B_s) dB_s.
$$

It is readily seen that (for a subsequence if neccessary)

$$\int_{\mathbb{R}} b_n(y) dL_t^y(x+B.) \to \int_{\mathbb{R}} b(y) dL_t^y(x+B.)$$

μ almost surely. Using dominated convergence similar as for the first term we get

$$\lim_{n\to\infty} ii)_n = 0.$$

For the last term notice that $\mathcal{E}(\int_0^t b_n(x+B_u)dB_u) \to \mathcal{E}(\int_0^t b(x+B_u)dB_u)$ μ-almost surely (possibly for a subsequence. We note that $\{\mathcal{E}(\int_0^t b_n(x+B_u)dB_u)\}_{n\geq 1}$ is bounded in, say, $L^4(\Omega)$ as long as T is small enough. By uniform integrability we get

$$\|\mathcal{E}(\int_0^t b_n(x+B_u)dB_u) - \mathcal{E}(\int_0^t b(x+B_u)dB_u\|_{L^2(\Omega)} \to 0$$

as $n \to \infty$. Using dominated convergence we get

$$iii)_n \to 0,$$

which completes the proof.

The equality (17) is proved similary.

□

We now prove the main theorem:

Proof of 3.2. We let $\delta = T$ where T is as in Proposition 3.8. Let $t, s \in \mathbb{R}$ and let $k \in \mathbb{Z}$ be such that

$$(k-1)\delta \leq t-s < k\delta.$$

If k is positive, define

$$\phi_{s,t}(x) = \phi_{k\delta+s,t} \circ \phi_{(k-1)\delta+s,k\delta+s} \circ \cdots \circ \phi_{s,\delta s}(x),$$

and for negative k we define

$$\phi_{s,t}(x) = \phi_{t+k\delta,s} \circ \phi_{t+(k+1)\delta,t+k\delta} \circ \cdots \circ \phi_{t,t-\delta}(x).$$

It is readily checked that this is a solution to (3). □

4 Two Examples

In this section we will consider two specific examples of irregular drift coefficients which fit into the previous results. However, it is shown here that the solutions to these equations actually have a *continuously* differentiable flow.

We shall need two preliminary results.

Lemma 4.1. *Let $\alpha \in (0,1)$ be given. Then there exists a $T_\alpha > 0$ such that for every $t \in [0, T_\alpha]$ the solution X_t^x has a Hölder continuous version of exponent α in x on bounded sets.*

Proof. For $t = 0$ this is obvious, so assume $t > 0$. We know that $X_t^{n,x} \to X_t^x$ in $L^2(\Omega)$ and we may extract a subsequence which converges μ-a.s. (still denoted by n). Let $k \in \mathbb{N}$ be such that $\frac{k-1}{k} > \alpha$ and choose T_k such that

$$\sup_n E[|\phi'_{n,t}(x)|^k] \leq t^{-1/2} e^{cx^2}$$

for some constant c. Since $x \mapsto X_t^{n,x}$ is continuously differentiable we have

$$\begin{aligned} E[|\phi_{n,t}(x) - \phi_{n,t}(y)|^k] &= E[| \int_0^1 \phi'_{n,t}(\theta x + (1-\theta)y) d\theta|^k] |x-y|^k \\ &\leq \int_0^1 t^{-1/2} e^{c(\theta x + (1-\theta)y)^2} d\theta |x-y|^k. \end{aligned}$$

Letting n tend to infinity and applying Fatou's lemma we get

$$E[|\phi_t(x) - \phi_t(y)|^k] \leq C(t,x,y)|x-y|^k.$$

By Kolmogorov's lemma we get the result with $T_\alpha = T_k$. $\square$

Lemma 4.2. *Let $-\infty \leq a < b \leq \infty$. For every $\gamma < 1/2$ there exists an interval $[0,T]$ such that there exists a Hölder continuous version of the mapping*

$$x \mapsto \int_0^t 1_{(a,b]}(X_u^x) dB_u$$

with exponent γ. Similary for every $\gamma < 1$ there exists a Hölder continuous version of the mapping

$$x \mapsto \int_0^t 1_{(a,b]}(X_u^x) du$$

with exponent γ.

Proof. We begin by noting that there exists a constant C_n such that

$$E[| \int_0^t 1_{(u,v]}(X_s^x) ds|^n] \leq C_n |v-u|^n \tag{19}$$

for every $u, v \in \mathbb{R}$. Indeed, we have

$$E[|\int_0^t 1_{(u,v]}(X_s^x)ds|^n] = E[|\int_u^v L_t^y(X_\cdot^x)dy|^n]$$
$$\leq (v-u)^{n-1}\int_u^v E[|L_t^y(X_\cdot^x)|^n]dy$$

and since b is sub-linear we have by the BGD-lemma that

$$E[|L_t^y(X_\cdot^x)|^n] \leq C_n(E[|X_t^x - x|^n] + t^{n-1}\int_0^t (k_1 + k_2|X_s^x|)^k ds + t^{n/2}),$$

where the right-hand side is independent of y. Inequality (19) follows.

Let $\alpha \in (0,1)$. Let T be as in Lemma 4.1. For $K \in \mathbb{N}$ define the stopping time

$$\tau_K = \inf\{t > 0 : \|X_t^\cdot\|_{C^\alpha} > K\}.$$

By the BGD-lemma

$$\begin{aligned}
E[(\int_0^{t\wedge\tau_K} \left(1_{(a,b]}(X_u^x) - 1_{(a,b]}(X_u^y)\right) dB_u)^{2n}] \leq & C_1 E[(\int_0^{t\wedge\tau_K} (1_{(a,b]}(X_u^x) - 1_{(a,b]}(X_u^y))^2 du)^n] \\
\leq & C_2 \Bigg(E[(\int_0^{t\wedge\tau_K} 1_{\{X_u^x\in(a,b], X_u^y\leq a\}} du)^n] \\
& + E[(\int_0^{t\wedge\tau_K} 1_{\{X_u^x\in(a,b], X_u^y> b\}} du)^n] \\
& + E[(\int_0^{t\wedge\tau_K} 1_{\{X_u^y\in(a,b], X_u^x\leq a\}} du)^n] \\
& + E[(\int_0^{t\wedge\tau_K} 1_{\{X_u^y\in(a,b], X_u^x> b\}} du)^n] \Bigg).
\end{aligned}$$

For the first term, since $X_u^x - K|x-y|^\alpha \leq X_u^y$, we get

$$1_{\{X_u^x\in(a,b], X_u^y\leq a\}} \leq 1_{\{X_u^x\in(a,b], X_u^x - K|x-y|^\alpha \leq a\}}$$

and

$$E[(\int_0^{t\wedge\tau_K} 1_{\{a<X_u^x\leq b\wedge(a+K|x-y|^\alpha)\}} du)^n] \leq C_{K,n}|x-y|^{n\alpha}$$

from (19). The other terms are dealt with similarly. This leads to

$$E[(\int_0^{t\wedge\tau_K} 1_{(a,b]}(X_u^x) - 1_{(a,b]}(X_u^y) dB_u)^{2n}] \leq \tilde{C}_{K,n}|x-y|^{n\alpha}.$$

For a given $\gamma \in (0, \frac{1}{2})$ we see that we can choose α and n such that we may invoke Kolmogorov's lemma to get the result.

The second assertion is proved similarly.

$\square$

4.1 Step functions

In this section we will consider the case where b is a step function. More precisely, we assume that there are real numbers $-\infty < y_1 < \cdots < y_N < \infty$ and $b_1, \dots b_N \in \mathbb{R}$ such that

$$b(y) = \sum_{i=1}^{N-1} b_i 1_{(y_i, y_{i+1}]}(y). \tag{20}$$

In particular, we have $k_1 = \max_{1 \leq i \leq N} |b_i|$ and $k_2 = 0$ in (1). From Proposition 3.11 we get that the corresponding solution of (2) is weakly differentiable and

$$\begin{aligned} \frac{d}{dx} X_t^x &= \exp\{-\int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^x)\} \\ &= \exp\{-\sum_{i=1}^{N-1} b_i (L_t^{y_{i+1}}(X_\cdot^x) - L_t^{y_i}(X_\cdot^x))\}. \end{aligned}$$

Recall that the local time of $X_\cdot^x$ (at the point y) can be defined as the following process

$$L_t^y(X_\cdot^x) = |X_t^x - y| - |x - y| - \int_0^t sgn(X_s^x - y) dX_s^x. \tag{21}$$

Let $0 < \alpha < 1$. By Lemma 4.1 we can pick a version of X_t^x of which every trajectory is Hölder continuous in x of order α. We then see that the first term in the right hand side of (21) is also C^α. The second term is obviously Lipschitz in x.

For the remaining term, we write

$$\int_0^t sgn(X_s^x - y) dX_s^x = \sum_{i=1}^{N-1} b_i \left(\int_0^t sgn(X_s^x - y) 1_{(y_i, y_{i+1}]}(X_s^x) ds + \int_0^t sgn(X_s^x - y) dB_s \right). \tag{22}$$

If we now write

$$\int_0^t sgn(X_s^x - y) dB_s = \int_0^t 1_{(y,\infty]}(X_s^x) dB_s - \int_0^t 1_{(-\infty,y]}(X_s^x) dB_s$$

and

$$\int_0^t sgn(X_s^x - y) 1_{(y_i, y_{i+1}]}(X_s^x) ds$$

$$= \begin{cases} \int_0^t 1_{(y_i, y_{i+1}]}(X_s^x) ds & , y > y_{i+1} \\ -\int_0^t 1_{(y_i, y_{i+1}]}(X_s^x) ds & , y \leq y_i \\ -\int_0^t 1_{(y_i, y]}(X_s^x) ds + \int_0^t 1_{(y, y_{i+1}]}(X_s^x) ds & , y \in (y_i, y_{i+1}] \end{cases},$$

we get from (22) and (21) that there exists a version of $x \mapsto L_t^y(X_\cdot^x)$ which is Hölder continuous of order $\alpha < 1/2$.

Combining the above with the explicit representation of the spatial derivative of X_t^x we can summarize:

Theorem 4.3. *Assume b is a step function of the form (20). For any number $0 < \alpha < 1.5$ the corresponding solution X_t^x to (2) has a version which is Hölder continuous of order α in x. In particular, the mapping $x \mapsto X_t^x$ is continuously differentiable.*

4.2 Continuous and Unbounded drift

In this section we consider the drift $b(y) = y1_{[c,\infty)}(y)$ where $c \in \mathbb{R}$. By Proposition 3.4 we know that there exists an interval $[0, T]$ and a solution on this interval:

$$X_t^x = x + \int_0^t X_u^x 1_{[c,\infty)}(X_u^x)du + B_t,$$

which is weakly differentiable in x and Malliavin differentiable in ω.

Moreover, we have

$$\frac{d}{dx}X_t^x = \exp\{cL_t^c(X_\cdot^x) + \int_0^t 1_{[c,\infty)}(X_u^x)du\} \tag{23}$$

for Lebesgue almost every x and μ-a.s. The Malliavin derivative can be expressed

$$D_s X_t^x = \exp\{c(L_t^c(X_\cdot^x) - L_s^c(X_\cdot^x)) + \int_s^t 1_{[c,\infty)}(X_u^x)du\}$$

To see this we note that by Proposition 3.11 it is enough to prove that

$$-\int_{\mathbb{R}} y1_{[c,\infty)}(y)dL_t^y(X_\cdot^x) = \int_0^t 1_{[c,\infty)}(X_u^x)du + cL_t^c(X_\cdot^x).$$

We start by noting that $y \mapsto L_t^y(X_\cdot^x)$ has a continuous modification. Indeed, by [15], Chapter VI. Local Times, we get that there exists a cadlag modification of $y \mapsto L_t^y(X_\cdot^x)$ and we have

$$L_t^y(X_\cdot^x) - L_t^{y-}(X_\cdot^x) = 2\int_0^t 1_{\{X_s=y\}}b(X_s)ds = 0.$$

Fix x and $\Omega_0 \subset \Omega$ with full measure such that $y \mapsto L_t^y(X_\cdot^x)$ is continuous and the equalities (15) and (17) hold on Ω_0.

Define an approximating sequence

$$b_n(y) = \begin{cases} 0 & , y \leq c - \frac{1}{n} \\ ncy + c(1-nc) & , c - \frac{1}{n} < y < c \\ y & , c \leq y \end{cases}.$$

Then each b_n is differentiable almost everywhere and $b_n(y) \to b(y)$ for every $y \in \mathbb{R}$.

It can be verified that

$$\int_0^t b_n(X_s^x) dX_s^x \to \int_0^t b(X_s^x) dX_s^x, \; \mu - a.s.$$

as $n \to \infty$, and so that

$$\begin{aligned} \int_{\mathbb{R}} b_n(y) dL_t^y(X_\cdot^x) =& 2\int_0^1 b_n(\theta X_t^x - (1-\theta)x) d\theta (X_t^x - x) - 2\int_0^t b_n(X_u^x) dX_u^x \\ \to& 2\int_0^1 b(\theta X_t^x - (1-\theta)x) d\theta (X_t^x - x) - 2\int_0^t b(X_u^x) dX_u^x \\ =& \int_{\mathbb{R}} b(y) dL_t^y(X_\cdot^x), \; \mu - a.s. \end{aligned}$$

as $n \to \infty$. Furthermore, we notice that since b_n is almost everywhere differentiable we have

$$\begin{aligned} -\int_{\mathbb{R}} b_n(y) dL_t^y(X_\cdot^x) &= \int_{\mathbb{R}} b_n'(y) L_t^y(X_\cdot^x) dy \\ = \int_{c-n^{-1}}^c ncL_t^y(X_\cdot^x) dy &+ \int_{\mathbb{R}} 1_{[c,\infty)}(y) L_t^y(X_\cdot^x) dy. \end{aligned}$$

The last term can be rewritten

$$\int_{\mathbb{R}} 1_{[c,\infty)}(y) L_t^y(X_\cdot^x) dy = \int_0^t 1_{[c,\infty)}(X_u^x) du$$

and by the continuity of $y \mapsto L_t^y(X_\cdot^x)$ the first term converges to $cL_t^c(X_\cdot^x)$.

Finally, we show that $x \mapsto \frac{d}{dx} X_t^x$ actually has a continuous modification using the representation (23).

Continuity of $x \mapsto \int_0^t 1_{[c,\infty)}(X_u^x) du$ follows by a similar argument as in Section 4.1.

To see that $L_t^c(X_\cdot^x)$ is continuous in x we rewrite

$$\begin{aligned} L_t^c(X_\cdot^x) &= |X_t^x - c| - |x - c| - \int_0^t sgn(X_s^x - c) 1_{[c,\infty)}(X_s^x) X_s^x ds - \int_0^t sgn(X_s^x - c) dB_s \\ &= |X_t^x - c| - |x - c| - \int_0^t 1_{[c,\infty)}(X_s^x) X_s^x ds - \int_0^t sgn(X_s^x - c) dB_s \\ &= |X_t^x - c| - |x - c| - (X_t^x - x - B_t) - \int_0^t sgn(X_s^x - c) dB_s, \end{aligned}$$

since X_t^x solves the SDE. Similar as above we can prove that $x \mapsto \int_0^t sgn(X_s^x - c) dB_s$ has a continuous version and likewise for $x \mapsto L_t^c(X_\cdot^x)$.

Then we see that

$$\frac{d}{dx} X_t^x = \exp\{cL_t^c(X_\cdot^x) + \int_0^t 1_{[c,\infty)}(X_u^x) du\}$$

has a continuous modification.

5 Stochastic Transport Equation

In this section we will apply the results of the previous section to obtain continuously differentiable solutions of the Stochastic Transport Equation. This section will not include detailed proofs as the result is built on the existing result found in [12].

The stochastic transport equation is the following equation

$$\begin{cases} d_t u(t,x) + & b(x)\frac{\partial}{\partial x}u(t,x)dt + \frac{\partial}{\partial x}u(t,x) \circ dB_t = 0 \\ u(0,x) = & u_0(x), \end{cases} \tag{24}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a given function and $u_0 : \mathbb{R} \to \mathbb{R}$ is a given initial data. The stochastic integration is understood in the Stratonovich sense.

By a continuously differentiable, weak L^∞-solution of the transport equation (24) we mean a stochastic process $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R})$ such that, for every t, the function $u(t,\cdot)$ is continuously differentiable a.s. with $E[\left|\frac{\partial}{\partial x}u(t,x)\right|^4] < \infty$ and for every test function $\theta \in C_0^\infty(\mathbb{R})$, the process $\int_{\mathbb{R}} \theta(x)u(t,x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semi martingale and

$$\begin{aligned} \int_{\mathbb{R}} \theta(x)u(t,x)dx = & \int_{\mathbb{R}} \theta(x)u_0(x)dx \\ & - \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}u(s,x)b(x)\theta(x)dxds \\ & + \int_0^t \left(\int_{\mathbb{R}} u(s,x)\theta'(x)dx \right) \circ dB_s. \end{aligned} \tag{25}$$

Remark 5.1. *Note that the Stratonovich integral may be written*

$$\int_0^t \left(\int_{\mathbb{R}} u(s,x)\theta'(x)dx \right) \circ dB_s$$
$$= \int_0^t \left(\int_{\mathbb{R}} u(s,x)\theta'(x)dx \right) dB_s + \frac{1}{2}\int_0^t \left(\int_{\mathbb{R}} u(s,x)\theta''(x)dx \right) ds.$$

Since we don't know if $u(s,\cdot)$ is twice differentiable, we cannot use integration by parts in the last term. Thus, we need to integrate agains test-functions for this definition.

Theorem 5.2. *Let b be a step function as in (20), and let $u_0 \in C_b^1(\mathbb{R}^d)$. Then there exists a unique continuously differentiable, weak L^∞-solution $u(t,x)$ to (24). Moreover, for fixed t and x, this solution is Malliavin-differentiable.*

Proof. Since b is bounded, we get by [12] that (24) is uniquely solved by the weakly differentiable funtion

$$u(t,x) = u_0(\phi_t^{-1}(x)).$$

By Theorem 4.3 we know that there exists a version of $x \mapsto \phi_t^{-1}(x)$ which is continuously differentiable. This proves the claim. $\square$

6 Appendix

The following result which is due to [1] provides a compactness criterion for subsets of $L^2(\mu)$ using Malliavin calculus. See e.g. [13] for more information about Malliavin calculus.

Theorem 6.1. *Let $\{(\Omega, \mathcal{A}, P)\,; H\}$ be a Gaussian probability space, that is $(\Omega, \mathcal{A}, P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ-field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \ldots, h_n)) = \sum_{i=1}^{n} \partial_i f(h_1, \ldots, h_n) h_i, \;\; h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega;H)}\,.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \left\|C^{-1}\mathbf{D}\, G\right\|_{L^2(\Omega;H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [1].

Lemma 6.2. *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0,1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0,1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt\, dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem 6.1 and Lemma 6.2 is now the following compactness criterion which is essential for the proof of Corollary 3.7 and Proposition 3.4:

Corollary 6.3. *Let $X_n \in \mathbb{D}^{1,2}$, $n = 1, 2...$, be a sequence of $\mathcal{F}_1$-measurable random variables such that there are constants $\beta > 0$ and $C > 0$ with*

$$\sup_n \|X_n\|_{L^2(\Omega)} \leq C$$

$$\sup_n \int_0^1 \int_0^1 \frac{E\left[\|D_t X_n - D_{t'} X_n\|^2\right]}{|t - t'|^{1+2\beta}} dt dt' \leq C$$

and

$$\sup_n \int_0^1 E\left[\|D_t X_n\|^2\right] dt \leq C.$$

where D_t denotes Malliavin differentiation. Then the sequence X_n, $n = 1, 2...$, is relatively compact in $L^2(\Omega)$.

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